Introduction to Gauge theory

1. Bundles, connections, Chern classes.

2. Gauge theory, holomorphic bundles, Donaldson theory.

3. Higher dimensional gauge theory, calibrations.

4. Seiberg-Witten, Bauer-Furuta.

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Pre-requisites

Manifolds, complex manifolds, differential forms, de Rham cohomology, basic Hodge theory:

$$\begin{aligned} H^* &= \frac{\ker d}{\operatorname{im} d} \qquad d \in \{\partial, d, \bar\partial\} \\ &\cong \ker d \cap (\operatorname{im} d)^\perp \\ &= \ker d \cap \ker d^* \\ &= \ker \Delta \qquad (\Delta = dd^* + d^*d) \\ &= \mathcal{H}. \end{aligned}$$

Vector bundles

We start with a C-manifold M, where C is one of {continuous, C^1, C^∞, C^ω, holomorphic, algebraic}, i.e. its transition functions on overlaps are of class C, making it possible to define functions of class C on M.

A vector bundle over M is a family "C-varying" vector spaces over M; i.e. each fibre has a linear structure – can add, scalar multiply, etc.

Definition 1 *A rank r complex vector bundle E over M is a C-manifold $E \xrightarrow{\pi} M$ such that there exist local isomorphisms g_U over small open subsets $U \subset M$ to $U \times \mathbb{C}^r \to U$ which differ on overlaps $UV := U \cap V$ by fibrewise-linear C-isomorphisms, i.e. by*

$$g_{UV} := g_U|_{UV} \circ g_V^{-1}|_{UV} : U \cap V \to GL(r, \mathbb{C}).$$

So we can form the bundle as $\coprod_U (U \times \mathbb{C}^r)/\sim$ where we glue via the transition functions g_{UV}: for $x \in U \cap V$ identify

$$U \times \mathbb{C}^r \ni \quad (x, v) \sim (x, g_{UV}(x)(v)) \quad \in V \times \mathbb{C}^r$$

for all $v \in \mathbb{C}^r$.

Then we can talk about C-sections; maps $s \colon M \to E$ such that $\pi \circ s = \mathrm{id}$ of class C in local trivialisations. The vector space of sections is denoted $\Gamma(E)$.

E is called *trivial* if is isomorphic to a product $M \times \mathbb{C}^r$; equivalently if it has r global sections which form a basis of the fibre $E_x := \pi^{-1}(x)$ at each point $x \in M$.

Bundles on contractible manifolds are trivial (lift a contraction of M to x to a contraction of E to E_x, and pick a trivialisation of E_x) in the classes $C = C^r$, C^∞. Homotopic bundles are isomorphic.

From now on $C = C^\infty$ unless specifically mentioned.

Example – bundles on spheres

A bundle over $S^n = D^n \cup_{S^{n-1}} D^n$ is trivial over each hemisphere D^n, and glued across S^{n-1} (or small neighbourhood $U \cap V$ thereof – the overlap of open sets U, V containing the hemispheres) by a map

$$S^{n-1} \to GL(r, \mathbb{C}).$$

Thus isomorphisms classes of bundles on S^n are in 1-1 correspondence with $\pi_{n-1}(GL(r, \mathbb{C}))$.

For instance line bundles L on $S^2 = \mathbb{P}^1$:

$$\pi_1(GL(1, \mathbb{C})) = \pi_1(\mathbb{C}^*) = \pi_1(U(1)) \cong \mathbb{Z};$$

we call this integer classifying the bundle its *first Chern class* $c_1(L)$.

This bundle can be constructed holomorphically since there is a holomorphic representative of $n \in \mathbb{Z} \cong \pi_1(\mathbb{C}^*)$ given by the map $z^n \colon U \cap V \to \mathbb{C}^*$. This holomorphic bundle is denoted $H^n = H^{\otimes n}$ or $\mathcal{O}(n)$.

We can relate $n = c_1$ to zeros of sections. Pick the trivial section 1 in the trivialisation of L over U; under the transition map z^n this becomes the section z^n in $L|_V$'s trivialisation over $U \cap V$. So we can extend this to a section of L over all of S^2 having n zeros in V.

So $c_1(L) = e(L)$ is the number of (signed) zeros of a section of L – the self intersection of the zero section S^2 in L.

Chern classes

More generally we could define $c_1(L) \in H^2(M, \mathbb{Z})$ for any complex line bundle L over any compact manifold M to be the Poincaré dual of the zero set of a transverse C^∞ section.

(Zero set a submanifold; given another section can choose a transverse isotopy between the two whose zero set gives a cobordism between the two zero sets making them homologous.)

We would also like to define $c_p(E) \in H^{2p}(M)$ to be Poincaré dual to the dependency locus of $(r-p+1)$ generic sections of E,

$$c_p(E) = \mathrm{PD}(Z(s_1 \wedge \ldots \wedge s_{r-p+1})) \in H^{2p}(M).$$

(So $c_r(E) = e(E),\ c_1(E) = c_1(\Lambda^r E)$.)

Then, choosing sections l_i of L_i, we find (on dropping the PDs for clarity),

$$\begin{aligned} c_2(L_1 \oplus L_2) &= Z(l_1 \oplus l_2) = Z(l_1) \cap Z(l_2) \\ &= c_1(L_1) \cup c_1(L_2). \end{aligned}$$

Similarly, for E of rank 2 with sections s_i,

$$\begin{aligned} c_2(E \oplus L) &= Z((s_1 \oplus 0) \wedge (s_2 \oplus l)) \\ &= Z((s_1 \wedge s_2) \oplus (s_1 \otimes l)) \\ & \quad \text{(via } \Lambda^2(E \oplus L) = \Lambda^2 E \ \oplus \ E \otimes L) \\ &= Z(s_1 \wedge s_2) \cap Z(l) \ \cup \ Z(s_1) \\ &= c_1(E)c_1(L) + c_2(E). \end{aligned}$$

More generally, for bundles A, B of ranks a, b, we have the decomposition

$$\Lambda^p(A \oplus B) \cong \bigoplus_{i=0}^{p} \Lambda^i(A) \otimes \Lambda^{p-i}(B).$$

So choosing p sections $(a_i \oplus b_i)$, we compute $c_{r-p+1}(A \oplus B)$ ($r = a + b = \mathrm{rk}(A \oplus B)$) via

$$\begin{aligned} Z\Big(&(a_1 \wedge \ldots \wedge a_p) \oplus (a_1 \wedge \ldots \wedge a_{p-1} \otimes b_p) \oplus \ldots \\ &\ldots \oplus (a_1 \otimes b_2 \wedge \ldots \wedge b_p) \oplus (b_1 \wedge \ldots b_p)\Big). \end{aligned}$$

This equals

$$
\begin{aligned}
& \Big(Z(a_1) \cap Z(b_1 \wedge \ldots \wedge b_n)\Big) \\
\cup \quad & \Big(Z(a_1 \wedge a_2) \cap Z(b_2 \wedge \ldots \wedge b_p)\Big) \\
\cup \ \ldots \ \cup \quad & \Big(Z(a_1 \wedge \ldots \wedge a_p) \cap Z(b_p)\Big).
\end{aligned}
$$

Taking Poincaré duals, then,

$$
\begin{aligned}
& c_{r-p+1}(A \oplus B) = \\
& c_a(A) c_{b-p+1}(B) + \ \ldots \ + c_{a-p+1}(A) c_b(B),
\end{aligned}
$$

giving the *Whitney sum formula*

$$c_n(A \oplus B) = \sum_{p+q=n} c_p(A) c_q(B).$$

Letting $c_\bullet(E) = \sum_i c_i(E) \in H^{2\bullet}(M)$ ($c_0 \equiv 1$) denote the *total Chern class* we can write this

$$c_\bullet(A \oplus B) = c_\bullet(A) c_\bullet(B).$$

Defining Chern classes in cohomology instead of homology gives them functoriality $c_\bullet(f^*E) = f^*c_\bullet(E)$ and allows them to be defined for C^0-vector bundles on more general topological spaces. In fact the Whitney sum formula, $c_0 = 1$, $c_1(\mathcal{O}(1))$ on $\mathbb{P}^n$ being the standard generator $[\omega] \in H^2(\mathbb{P}^n, \mathbb{Z})$, and functoriality are enough to determine the Chern classes generally.

Any bundle is a quotient of a trivial infinite dimensional bundle

$$\Gamma(E) \times M \xrightarrow{\text{ev}} E \to 0,$$

defining a map $f\colon M \to \mathrm{Gr}(\infty, r)$ to the infinite Grassmannian, unique up to homotopy. The cohomology of Gr can be calculated to be a polynomial ring on generators $c_i \in H^{2i}$ (the Chern classes of the universal quotient bundle Q!) and we *define* the Chern classes of E by

$$c_i(E) = c_i(f^*Q) := f^*c_i \quad (= f^*c_i(Q)).$$

We now want to see these classes on manifolds defined in terms of differential geometry.

Connections

Connections give a way of differentiating in a bundle; equivalently a choice of specifying which sections are constant ("parallel").

Definition 2 *A connection A in the bundle E is a $\mathbb{C}$-linear map*

$$\begin{aligned} & d_A \colon \Gamma(E) \to \Omega^1(E) \\ \textit{s.t.} \quad & d_A(fs) = s df + f d_A s \quad \forall f \in C^\infty(M). \end{aligned}$$

I.e. d_A is a local (but not tensorial; i.e. not $C^\infty(M)$-linear) operator $T_xM \otimes \Gamma(E) \to E_x$ such that $(d_A)_X(fs) = s.X(f) + f.(d_A)_X s$.)

This extends uniquely to $d_A \colon \Omega^p(E) \to \Omega^{p+1}(E)$ such that

$$d_A(\omega \wedge s) = d_A\omega \wedge s + (-1)^i \omega \wedge d_A s$$

for all $\omega \in \Omega^i(M)$, $s \in \Omega^p(E)$.

Connections are far from unique; the difference of 2 connections is a tensor in $\Omega^1(\operatorname{End} E)$:

$$\begin{aligned}(d_A - d_B)(fs) &= sdf + fd_A s - (sdf + fd_B s) \\ &= f(d_A - d_B)s\end{aligned}$$

so $d_A - d_B$ is $C^\infty(M)$-linear.

The converse is also true: $d_A + a\colon s \mapsto d_A s + a(s)$ is a connection. So in a local trivialisation connections just look like $d+a$ so by gluing a's by a partition of unity argument we can show they exist.

Therefore the set of connections is

$$\{d_{A_0} + a \,:\, a \in \Omega^1(\operatorname{End} E)\},$$

an infinite dimensional affine space modelled on $\Omega^1(\operatorname{End} E)$.

Given any $\sigma_x \in \Omega^1(E)_x$ we can find a section s of E such that $s(x) = 0$, $(d_A s)_x = \sigma_x$.
(The question is local so we can assume that E is trivial over $U \subset \mathbb{R}^n$ but $d_A = d + a$ may be non-trivial: $a \neq 0$. Then $\Omega^1(E)_x = T^*_x M \otimes E_x$ is invariantly the E-valued linear functions vanishing at x, so σ_x defines a linear section s such that $s(x) = 0$ and $(ds)_x = \sigma_x$. Thus $d_A(s)_x = (ds)_x + a(s)_x = \sigma_x + 0 = \sigma_x$.)

So given an element of E_x can pick a section e with that value at x whose derivative at x is zero. (Set $\sigma_x = (d_A e)_x$ giving s as above; then replace e by $e + s$.)

Differentiating this section (thought of as a map $M \to E$) gives $\pi^* TM \to TE$ splitting the exact sequence of bundles

$$0 \to T_\pi E \to TE \to \pi^* TM \to 0.$$

An equivalent definition of a connection is such a splitting.

Curvature

Can in fact integrate up such horizontal lifts e to give parallel sections along curves in M.

Parallel transporting a frame of E_x around an infinitesimally small loop in M gives an infinitesimal automorphism of the fibre E_x; i.e. an element of $\operatorname{End} E_x$; the *curvature* of A at x.

$$F_A := d_A^2 \colon \Gamma(E) \to \Omega^2(E)$$

is a *tensor* $F_A \in \Omega^2(\operatorname{End} E)$:

$$\begin{aligned} d_A^2(fs) &= d_A(f d_A s + s df) \\ &= df \wedge d_A s + f d_A^2 s + d_A s \wedge df \\ &= f d_A^2 s. \end{aligned}$$

Tensors can be differentiated with respect to A by the obvious product rule

$$(d_A h)(s_1, \ldots, s_k) := d(h(s_1, \ldots, s_k))$$
$$-h(d_A s_1, s_2, \ldots, s_k) - \ldots - h(s_1, \ldots, s_{k-1}, d_A s_k).$$

Thus we find that

$$\begin{aligned} F_{A+a} &= (d_A + a)(d_A + a) \\ &= d_A^2 + d_A(a \wedge \cdot) + a \wedge d_A + a \wedge a \\ &= F_A + d_A a + a \wedge a. \end{aligned}$$

Also (exercise)

$$d_A F_A = 0,$$

the (second) Bianchi identity.

Metrics

A hermitian metric h on E is a C^∞ choice of a hermitian metric $h(\,\cdot\,,\,\cdot\,)_x$ on each fibre E_x, i.e. a section $h \in \Gamma(E^* \otimes \bar{E}^*)$ which is conjugate-symmetric and nondegenerate on each fibre.

Using convexity of the space of hermitian metrics and partitions of unity we can glue local metrics (positive definite self-adjoint matrices $h_{ij} = h(s_i, s_j) = \overline{h_{ji}}$) to give global ones.

A *unitary* connection is one for which $d_A h = 0$. Picking A_0 unitary we find that the space of unitary connections $\mathcal{A}$ is

$$\mathcal{A} := A_0 + \Omega^1(\mathfrak{u}(E)),$$

where $\mathfrak{u}$ denotes skew-adjoint endomorphisms. Accordingly parallel transport is unitary and $F_A \in \Omega^2(\mathfrak{u}(E))$. (All bundles, connections unitary from now on unless specified.)

Automorphisms of bundles are called, intimidatingly, *gauge transformations*. If A is a connection and g an automorphism, then we can form the pull-back connection g^*A by

$$g^{-1} \circ d_A \circ g = d_A + g^{-1} d_A(g).$$

The group of gauge transformations is usually denoted by $\mathcal{G}$, and the set of isomorphism classes of connections, $\mathcal{A}/\mathcal{G}$, by $\mathcal{B}$.

Chern-Weil theory

From $d_A F_A = 0$ it follows that $d\,\mathrm{tr}\,F_A = \mathrm{tr}\,d_A F_A = 0$ so that $[\mathrm{tr}\,F_A] \in H^2(M, \mathbb{R})$. If we replace A by $A + a$ then we get

$$\begin{aligned} \mathrm{tr}(F_{A+a}) &= \mathrm{tr}(F_A + d_A a + a \wedge a) \\ &= \mathrm{tr}\,F_A + d\,\mathrm{tr}(a) + \mathrm{tr}(a \wedge a). \end{aligned}$$

Since $tr(AB)$ is symmetric in A, B and $\wedge$ is antisymmetric, the last term vanishes. So we find that $[\mathrm{tr}\,F_A] \in H^2(M, \mathbb{R})$ is independent of the choice of A. What is it ?

We go back to our example of $\mathcal{O}(n) \to S^2 = D^2 \cup_{S^1} D^2$. Pick a trivialisation of $\mathcal{O}(n)$ over D_1, and pick the trivial connection on it. This trivialisation is then glued across the equator S^1 to a trivialisation over D_2 by any degree n function $f\colon S^1 \to \mathbb{C}^*$ (e.g. $f = z^n$).

Thinking of this function as a gauge transformation it takes the trivial connection on D_1 to the connection $d + f^{-1}df$ on D_2, in the trivialisation on D_2. We can extend this arbitrarily to a connection $A = d + a$ over D_2. Then

$$\begin{aligned}\int_{S^2} F_A &= \int_{D_2} F_A \\ &= \int_{D_2} da = \int_{S^1} a \\ &= \int_{S^1} d\log f = 2\pi i n.\end{aligned}$$

(E.g. using $f = z^n$, $a = d\log f = f^{-1}df = ndz/z$ and $\int_{S^1} ndz/z = 2\pi i n$.) It follows that

$$\left[\frac{\operatorname{tr} F_A}{2\pi i}\right] = c_1 \in H^2(M, \mathbb{Z})/\text{torsion}.$$

Similarly (exercise) all ad-invariant polynomials of $\operatorname{End} E$ (tr, det, $\operatorname{tr}(\ )^2$, etc.) applied to F_A give de Rham cohomology classes independent of the connection.

(The ad-invariant polynomials generate the cohomology of the Grassmannian.)

These also have integrality properties, and one can check that, modulo torsion, the following definition coincides (for manifolds) with the topological one.

Definition 3 $c_\bullet(E) := \det\left(\mathrm{id} + \frac{F_A}{2\pi i}\right)$, *i.e.*

$$1+c_1(E)+c_2(E)+\ldots = 1+\frac{\operatorname{tr} F_A}{2\pi i}-\frac{\operatorname{tr} F_A \wedge F_A}{4\pi^2}+\ldots$$

(So e.g. from $F_{\Lambda^r A} = \operatorname{tr} F_A$ we recover $c_1(\Lambda^r E) = c_1(E)$.)

Connections and holomorphic structures

A $\bar\partial$-operator in a bundle on a complex manifold X is "half" a connection – we can decree which sections are *holomorphic*, not constant.

Definition 4 *A $\bar\partial$-operator A in the bundle E is a $\mathbb{C}$-linear map*

$$\begin{aligned} & \bar\partial_A \colon \Gamma(E) \to \Omega^{0,1}(E) \\ \textit{s.t.} \quad & \bar\partial_A(fs) = s\bar\partial f + f\bar\partial_A s \quad \forall f \in C^\infty(X). \end{aligned}$$

Extends as before to $\Omega^{p,q}(E) \xrightarrow{\bar\partial_A} \Omega^{p,q+1}(E)$.

Since we are on a complex manifold, $\Omega_1 \otimes \mathbb{C} \cong \Omega^{1,0} \oplus \Omega^{0,1}$ and any connection A splits as $d_A = \partial_A \oplus \bar\partial_A$.

Any *holomorphic* bundle E has a *canonical* $\bar\partial$-operator $\bar\partial_E$ since we already know its kernel (the holomorphic sections) and the rest follows from the Leibniz rule. I.e. since locally any section is a C^∞-linear combination of local holomorphic sections $\{e_i\}_{i=1}^r$, $\bar\partial_E e_i = 0$, this determines

$$\bar\partial_E\left(\sum \alpha_i e_i\right) = \sum (\bar\partial \alpha_i).e_i\,.$$

(I.e. usual $\bar\partial$ on open sets; on overlaps we have holomorphic transition functions: $e_i' = \sum \phi_{ij} e_j$. But $\bar\partial \phi_{ij} = 0$ so we still have $\bar\partial_E e_i' = 0$, so $\bar\partial_E$ is well defined.)

So $\bar\partial_E^2 = 0$, and the *Newlander-Nirenberg theorem* gives the converse. $\bar\partial_A^2 = 0$ is the integrability condition for finding local bases of solutions of $\bar\partial_A s = 0$; these then define the local holomorphic trivialisation of the bundle, and transition functions between different patches are therefore holomorphic, defining a holomorphic structure on E.

Theorem 5 *A hermitian metric h on a holomorphic bundle E determines a unique unitary* Chern *connection d_A compatible with $\bar{\partial}_E$:*

$$d_A(h) = 0 \quad \textit{and} \quad \bar{\partial}_A = \bar{\partial}_E.$$

(C.f. the Levi-Civita connection on TM; unique connection compatible with metric and *torsion-free*.)

Proof. Picking a local holomorphic trivialisation $\{e_i\}_{i=1}^r$ s.t. $h = (h_{ij}) = h(s_i, s_j)$, then

$$\bar{\partial} h_{ij} = h(\bar{\partial}_E e_i, e_j) + h(e_i, \partial_A e_j) = h(e_i, \partial_A e_j)$$

uniquely determines $d_A e_i = \partial_A e_i$ as

$$\partial_A e_i = \sum_{jk} \partial h_{ij} (h^{-1})_{jk} e_k.$$

Conversely the Leibniz rule shows this determines a compatible connection d_A.

Alternatively, in a local unitary frame $\{e_i\}_{i=1}^r$, unitarity forces $d\,h(e_i, e_j) = 0$, i.e.

$$h(\partial_A e_i, e_j) = -h(e_i, \bar{\partial}_E e_j)$$

so determining $\partial_A e_i$ (and so d_A) from $\bar{\partial}_E$. $\square$

Connections on complex manifolds have curvature

$$\begin{aligned} F_A = d_A^2 &= F_A^{2,0} \oplus F_A^{1,1} \oplus F_A^{0,2} \\ &= \partial_A^2 \oplus (\partial_A\bar{\partial}_A + \bar{\partial}_A\partial_A) \oplus \bar{\partial}_A^2, \end{aligned}$$

and so are compatible with the (or define a) holomorphic structure if and only if $F_A^{0,2} = 0$ – a prototype of a gauge theory equation.

So given a holomorphic bundle, metric $\Rightarrow$ connection. To get a closer link try to fix metric by imposing an equation on the resulting curvature; e.g. the Hermitian-Yang-Mills equation. (Compare uniformisation for Riemann surfaces: can study complex geometry by introducing a metric; if we impose scalar curvature = constant (and volume=1) the metric is unique and the study of the complex geometry and (constant scalar curvature) Kähler geometry are equivalent. Similarly Yau's theorem for Hermitian-Einstein metrics in higher dimensions.)